

The non-commutative Khintchine inequalities for $0 < p < 1$

by

Gilles Pisier and Éric Ricard*

Texas A&M University and Université de Caen Basse-Normandie

Abstract

We give a proof of the Khintchine inequalities in non-commutative L_p -spaces for all $0 < p < 1$. These new inequalities are valid for the Rademacher functions or Gaussian random variables, but also for more general sequences, e.g. for the analogues of such random variables in free probability. We also prove a factorization for operators from a Hilbert space to a non commutative L_p -space, which is new for $0 < p < 1$. We end by showing that Mazur maps are Hölder on semifinite von Neumann algebras.

2000 MSC 46L51, 46L07, 47L25, 47L20

*Partially supported by ANR-2011-BS01-008-01.

The Khintchine inequalities for non-commutative L_p -spaces were first proved by Lust-Piquard in [16]. They play an important rôle in the recent developments in non-commutative Functional Analysis, and in particular in Operator Space Theory, see [24]. Just like their commutative counterpart for ordinary L_p -spaces, they are a crucial tool to understand the behavior of unconditionally convergent series of random variables, or random vectors, in non-commutative L_p ([27]). The commutative version is closely related to Grothendieck's Theorem (see [20, 21]). Moreover, in the non-commutative case, Random Matrix Theory and Free Probability provide further ground for applications of the non-commutative Khintchine inequalities. For instance, they imply the remarkable fact that the Rademacher functions (i.e. i.i.d. ± 1 -valued independent random variables) satisfy the same inequalities as the freely independent ones in non-commutative L_p for $p < \infty$. See [8] for a recent direct simple proof of the free version of these inequalities, which extend to $p = \infty$.

In the most classical setting, the non-commutative Khintchine inequalities deal with Rademacher series of the form

$$S = \sum_k r_k(t) x_k$$

where (r_k) are the Rademacher functions on the Lebesgue interval (or any independent symmetric sequence of random choices of signs) where the coefficients x_k are in the Schatten q -class S_q or in a non-commutative L_q -space associated to a semifinite trace τ . Let us denote simply by $\|\cdot\|_q$ the norm (or quasi-norm) in the latter Banach (or quasi-Banach) space, that we will denote by X . By Kahane's well known results, this series converges almost surely in norm iff it converges in $L_q(dt; X)$ (and in fact in $L_p(dt; X)$ for any $0 < p < \infty$, but for obvious reasons we prefer to work in the present context with $p = q$). Thus to characterize the almost surely norm-convergent series such as S , it suffices to produce a two sided equivalent of $\|S\|_{L_q(dt; X)}$, and this is precisely what the non-commutative Khintchine inequalities provide:

For any $0 < q < \infty$ there are positive constants α_q, β_q such that for any finite set (x_1, \dots, x_n) in $X = S_q$ (or $X = L_q(\tau)$) we have

$$\frac{1}{\beta_q} |||(x_k)|||_q \leq \left(\int \|S(t)\|_q^q dt \right)^{1/q} \leq \alpha_q |||(x_k)|||_q$$

where $|||(x_k)|||_q$ is defined as follows:

If $2 \leq q < \infty$

$$|||(x_k)|||_q = \max \left\{ \left\| \left(\sum x_k^* x_k \right)^{\frac{1}{2}} \right\|_q, \left\| \left(\sum x_k x_k^* \right)^{\frac{1}{2}} \right\|_q \right\}$$

and if $0 < q \leq 2$:

$$(0.1) \quad |||x|||_q \stackrel{\text{def}}{=} \inf_{x_k = a_k + b_k} \left\{ \left\| \left(\sum a_k^* a_k \right)^{\frac{1}{2}} \right\|_q + \left\| \left(\sum b_k b_k^* \right)^{\frac{1}{2}} \right\|_q \right\}.$$

Note that $\beta_q = 1$ if $q \geq 2$, while $\alpha_q = 1$ if $q \leq 2$ and the corresponding one sided bounds are easy. The difficulty is to verify the other side.

The case $1 < q < \infty$ is due to Lust-Piquard [16]. The case $q = 1$ was proved (in two ways) in [18], together with a new proof of $1 < q < \infty$. This also implied the fact (independently observed by Junge) that $\alpha_q = O(\sqrt{q})$ when $q \rightarrow \infty$, which yielded an interesting subGaussian estimate. Later on, Buchholz proved in [4] a sharp version valid when $q > 2$ is any even integer, the best α_q happens to be the same as in the commutative (or scalar) case.

The case $q < 2$ of the Khintchine inequalities has a more delicate formulation, but this case can be handled easily when $1 < q < 2$ using a suitable duality argument. The case $q = 1$ is

closely related to the “little non-commutative Grothendieck inequality” in the sense of [26] (first proved in [23]): actually, one of the proofs given for that case in [18] shows that it is essentially “equivalent” to it. More recently, Haagerup and Musat ([9]) gave a new proof that yields the best constant (equal to 2) for $q = 1$ for the complex analogue (namely Steinhaus random variables) of the Rademacher functions.

In [25] the first named author proved by an extrapolation argument that the validity of this kind of inequalities for some $1 < q < 2$ implies their validity for all $1 \leq p < q$, but the case $q < 1$ remained open. However, very recently the second named author noticed that the method proposed in [25] actually works in this case too. The latter method reduced the problem to a certain form of Hölder type inequality which could not be verified because the required ingredients (duality and triangular projection) became seemingly unavailable for $0 < q < 1$. In [25] a certain very weak form of the required Hölder type estimate was identified as sufficient to complete the case $q < 1$. It is this form that the second named author was able to establish by an a priori ultraproduct argument (see Remark 1.6). Although his argument failed to produce explicitly a quantitative estimate, it showed that some estimate does exist. The goal of this paper is to produce an explicit estimate, and a reasonably self-contained proof of the case $q < 1$. In fact, it turns out that a certain version of Hölder’s inequality (perhaps of independent interest) does hold, thus we can produce an explicit estimate, similar to the case $q \geq 1$ but with unexpected exponents. This inequality, namely (2.2) below, may prove useful in the theory of means developed in [12, 13].

In the rest of the paper we will consider only the case $0 < q \leq 2$. In that case, our inequalities reduce to this : There is β_q such that for any finite sequence (x_k) in an arbitrary non-commutative L_q -space, we have

$$(0.2) \quad |||x|||_q \leq \beta_q \left(\int \left\| \sum r_k(t) x_k \right\|_q^q dt \right)^{\frac{1}{q}}$$

where $|||x|||_q$ is as in (0.1).

In this paper, as in [25], we will show that the validity of (0.2) for some fixed q with $1 < q < 2$ implies its validity (with another constant) for all value of q in $(0, q)$ (and in particular for all q in $(0, 1)$). For that deduction the only assumption needed on (r_k) is its orthonormality in $L_2([0, 1])$. Thus our approach yields (0.2) also for more general sequences than the Rademacher functions. For instance, we may apply it to free Haar unitaries in the sense of [36] or to the “ $Z(2)$ -sequences” considered in [11].

In §3 we prove an extension to the case $0 < q < 1$ of the “little Grothendieck inequality”, i.e. a (Maurey type) factorization for bounded linear maps from a Hilbert space to a non-commutative L_p -space.

In §4 we extend to the case $0 < p < 1$ some of the results of [29] giving Hölder exponents for the Mazur map $M_{p,q} : L_p(\tau) \rightarrow L_q(\tau)$ given by

$$M_{p,q}(f) = f|f|^{\frac{p-q}{q}},$$

relative to a semifinite von Neumann algebra (M, τ) .

For convenience, we recall an elementary fact: if X is an L_p -space (commutative or not) and if $0 < p \leq 1$, the quasi-norm $\| \cdot \|$ of X satisfies the “ p -triangle inequality”:

$$(0.3) \quad \forall x, y \in X \quad \|x + y\|^p \leq \|x\|^p + \|y\|^p.$$

Actually, it will be convenient to invoke a consequence of the triangle inequality, valid, this time, for all $0 < p \leq \infty$:

$$(0.4) \quad \forall x, y \in X \quad \|x + y\| \leq \chi_p(\|x\| + \|y\|) \leq 2\chi_p \max\{\|x\|, \|y\|\},$$

where

$$\chi_p = \max\{2^{\frac{1}{p}-1}, 1\}.$$

1 The case $1 \leq q < 2$ from [25]

In this section, we review (and partly reproduce) the previous attempt from [25] to explain the contribution of the present paper.

Here, $L_2([0, 1])$ can be replaced by any non-commutative L_2 -space $L_2(\varphi)$ associated to a semifinite generalized (i.e. “non-commutative”) measure space, and (r_k) is then replaced by an orthonormal sequence (ξ_k) in $L_2(\varphi)$. Then the right-hand side of (0.2) is replaced by

$$\left\| \sum \xi_k \otimes x_k \right\|_{L_q(\varphi \otimes \tau)}.$$

More precisely, by a (semifinite) generalized measure space (N, φ) we mean a von Neumann algebra N equipped with a faithful, normal, semifinite trace φ . Without loss of generality, we may always reduce consideration to the σ -finite case. Throughout this paper, we will use freely the basics of non-commutative integration as described in [22] or [33, Chap. IX].

Let us fix another generalized measure space (M, τ) . The inequality we are interested in now takes the following form:

$$(K_q) \quad \begin{cases} \exists \beta_q \text{ such that for any finite sequence} \\ x = (x_k) \text{ in } L_q(\tau) \text{ we have} \\ |||x|||_q \leq \beta_q \left\| \sum \xi_k \otimes x_k \right\|_{L_q(\varphi \otimes \tau)} \\ \text{where } |||\cdot|||_q \text{ is defined as in (0.1).} \end{cases}$$

In the Rademacher case, i.e. when $(\xi_k) = (r_k)$, we refer to these as the non-commutative Khintchine inequalities.

We can now state the main result of [25] for the case $q \geq 1$.

Theorem 1.1. [25] *Let $1 < q < 2$. Recall that (ξ_k) is assumed orthonormal in $L_2(\varphi)$. Then $(K_q) \Rightarrow (K_p)$ for all $1 \leq p < q$.*

Here is a sketch of the argument in [25]. We denote

$$S = \sum \xi_k \otimes x_k.$$

Let \mathcal{D} be the collection of all “densities,” i.e. all f in $L_1(\tau)_+$ with $\tau(f) = 1$. Fix p with $0 < p \leq q$. Then we denote for $x = (x_k)$

$$(1.1) \quad C_q(x) = \inf \left\{ \left\| \sum \xi_k \otimes y_k \right\|_q \right\}$$

where $\|\cdot\|_q$ is the norm in $L_q(\varphi \otimes \tau)$ and the infimum runs over all sequences $y = (y_k)$ in $L_q(\tau)$ for which there is f in \mathcal{D} such that

$$x_k = (f^{\frac{1}{p}-\frac{1}{q}} y_k + y_k f^{\frac{1}{p}-\frac{1}{q}})/2.$$

Note that $C_p(x) = \|S\|_p$.

The proof of Theorem 1.1 is based on a variant of “Maurey’s extrapolation principle” (see [20]) This combines three steps: (here C', C'', C''', \dots are constants independent of $x = (x_k)$ and we wish to emphasize that here p remains fixed while the index q in $C_q(x)$ is such that $p < q \leq 2$).

Step 1. Assuming (K_q) we have

$$|||x|||_p \leq C' C_q(x).$$

Step 2.

$$C_2(x) \leq C'' |||x|||_p.$$

Actually the converse inequality also holds (up to a constant), see [25].

Step 3.

$$C_q(x) \leq C''' C_p(x)^{1-\theta} C_2(x)^\theta,$$

where θ is defined by $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$. The three steps put all together yield

$$|||x|||_p \leq C' C''' C_p(x)^{1-\theta} (C'' |||x|||_p)^\theta$$

and hence

$$|||x|||_p \leq C'''' C_p(x) = C'''' \|S\|_p.$$

Only the proof of Step 3 required $p \geq 1$ in [25]. Note that actually it suffices that, for some $0 < \theta' < 1$, we have

$$(1.2) \quad C_q(x) \leq C''' C_p(x)^{1-\theta'} C_2(x)^{\theta'},$$

and we will show that this essentially holds in §2, but with a rather surprising exponent given by

$$1 - \theta' = (1 - \theta) \frac{R}{2},$$

where R is any number such that $0 < R < p$. As will be explained below in Remark 1.5, the bound in (1.2) can be deduced from the following variant of Hölder's inequality, that will be proved in §2: There is a constant c such that

$$(1.3) \quad \forall x \in L_2(\tau) \quad \forall f \in \mathcal{D} \quad \|f^{\alpha(1-\theta)} x + x f^{\alpha(1-\theta)}\|_q \leq c \|f^\alpha x + x f^\alpha\|_p^{1-\theta'} \|x\|_2^{\theta'}.$$

When $p \geq 1$ this holds (see [25]) with $\theta' = \theta$. In the commutative case (or if there is only one term), when $\theta' = \theta$ this reduces to Hölder's inequality for L_p -norms (just write $f^{\alpha(1-\theta)} x = (f^\alpha x)^{1-\theta} x^\theta$ and recall $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$), so this holds (with $c = 2^\theta$) for $0 < p < \infty$.

When $p > 1$ the (complete) boundedness of the triangular projection on S_p implies

$$(1.4) \quad \max\{\|f^\alpha x\|_p, \|x f^\alpha\|_p\} \lesssim \|f^\alpha x + x f^\alpha\|_p,$$

from which (1.3) with $\theta' = \theta$ is immediate (see [14] or [25, 1.9 (iii)]). However this fails for $p \leq 1$, because, by a well known argument, such an estimate would imply conversely the boundedness of the triangular projection, which fails for $p \leq 1$.

When $p < 1$ we do not know whether (1.2) or (1.3) holds with $\theta' = \theta$.

Remark 1.2. In Theorem 1.1, the assumption that (ξ_k) is orthonormal in $L_2(\varphi)$ can be replaced by the following one: for any finite sequence $y = (y_k)$ in $L_2(M, \tau)$ we have

$$(1.5) \quad \left\| \sum \xi_k \otimes y_k \right\|_{L_2(\varphi \otimes \tau)} \leq \left(\sum \|y_k\|_2^2 \right)^{\frac{1}{2}}.$$

We will need the following fact. Results of this kind originate in Arazy and Friedman's memoir [1] and can also be found in Junge and Parcet's paper [14] (see also [13] for related inequalities).

Lemma 1.3 ([30]). *Let Q_j ($j = 1, \dots, n$) be mutually orthogonal projections in M and let λ_j ($j = 1, \dots, n$) be non-negative numbers. There is a constant C so that for any $1 \leq q \leq \infty$ and $\theta \in [0, 1]$, for any x in $L_q(\tau)$*

$$\frac{1}{C} \|x\|_{L_q(\tau)} \leq \left\| \sum_{i,j=1}^n \frac{\lambda_i^\theta + \lambda_j^\theta}{(\lambda_i + \lambda_j)^\theta} Q_i x Q_j \right\|_{L_q(\tau)} \leq C \|x\|_{L_q(\tau)}$$

Remark 1.4. If $0 < p \leq 2$, the converse inequality to (K_p) is valid assuming that $\varphi(1) = 1$ and $\xi_k \in L_2(N, \varphi)$ is orthonormal or satisfies (1.5). Indeed, for any $t \geq 0$ in $N \otimes M$ since $\frac{p}{2} \leq 1$ and $\varphi(1) = 1$, by the operator concavity of $t \mapsto t^{\frac{p}{2}}$ (see [3, p. 115-120]), we have $\|t\|_{\frac{p}{2}} \leq \|\mathbb{E}^M(t)\|_{\frac{p}{2}}$ and hence, if $S = \sum \xi_k \otimes x_k$, we have

$$\|S\|_p = \|S^* S\|_{\frac{p}{2}}^{\frac{1}{2}} \leq \|\mathbb{E}^M(S^* S)\|_{\frac{p}{2}}^{\frac{1}{2}} \leq \left\| \left(\sum x_k^* x_k \right)^{\frac{1}{2}} \right\|_p,$$

and similarly

$$\|S\|_p \leq \left\| \left(\sum x_k x_k^* \right)^{\frac{1}{2}} \right\|_p.$$

From this we easily deduce

$$\|S\|_p \leq \chi_p \|x\|_p.$$

where χ_p is as in (0.4).

Remark 1.5. To extend Theorem 1.1 to the case $0 < p < 1$ the difficulty lies in Step 3, or in proving a certain form of Hölder inequality. As mentioned in [25], actually a much *weaker* estimate allows to conclude:

It suffices to show that there is a function $\varepsilon \mapsto \delta(\varepsilon)$ tending to zero with $\varepsilon > 0$ such that when $f \in \mathcal{D}$ we have $(\alpha = \frac{1}{p} - \frac{1}{2} = \frac{1}{r})$ ($1 < q < 2$):

$$(1.6) \quad [\|x\|_2 \leq 1, \|f^\alpha x + x f^\alpha\|_p \leq \varepsilon] \Rightarrow \|f^{\alpha(1-\theta)} x + x f^{\alpha(1-\theta)}\|_q \leq \delta(\varepsilon).$$

When $1 < p < \infty$, we have by [25], $\delta(\varepsilon) \leq C_p \varepsilon^{1-\theta}$. This estimate can already be found in [1, Ch. 3] for Schatten classes. We will show that (1.6) can be substituted to Step 3.

Indeed, setting $w(\varepsilon) = 2\delta(\varepsilon)\varepsilon^{-1}$, by homogeneity this implies

$$\|f^{\alpha(1-\theta)} x + x f^{\alpha(1-\theta)}\|_q \leq \delta(\varepsilon) \max\{\|x\|_2, \varepsilon^{-1} \|f^\alpha x + x f^\alpha\|_p\} \leq \delta(\varepsilon) \|x\|_2 + \frac{w(\varepsilon)}{2} \|f^\alpha x + x f^\alpha\|_p.$$

Fix $\varepsilon' > 0$. Let y_k be such that $x_k = (f^\alpha y_k + y_k f^\alpha)/2$ with

$$\left\| \sum \xi_k \otimes y_k \right\|_2 < C_2(x)(1 + \varepsilon').$$

We will again denote $S = \sum \xi_k \otimes x_k$, and we set $Y = \sum \xi_k \otimes y_k$.

Let us assume that (M, τ) is M_n equipped with usual trace (the argument works assuming merely that f has finite spectrum). We will use the orthonormal basis for which f is diagonal with coefficients denoted by (f_i) . We have then

$$(y_k)_{ij} = 2(f_i^\alpha + f_j^\alpha)^{-1} (x_k)_{ij}.$$

Then the above inequality, with Y in place of x , yields

$$(1.7) \quad \|f^{\alpha(1-\theta)}Y + Yf^{\alpha(1-\theta)}\|_q \leq \delta(\varepsilon)\|Y\|_2 + w(\varepsilon)\|S\|_p.$$

We will now compare the elements T and Z defined by

$$T = f^{\alpha(1-\theta)}Y + Yf^{\alpha(1-\theta)} = [f_i^{\alpha(1-\theta)}Y_{ij} + Y_{ij}f_j^{\alpha(1-\theta)}] = 2\left[\frac{f_i^{\alpha(1-\theta)} + f_j^{\alpha(1-\theta)}}{f_i^\alpha + f_j^\alpha}x_{ij}\right],$$

$$Z = \left[\frac{2}{f_i^{\alpha\theta} + f_j^{\alpha\theta}}S_{ij}\right] = \left[\frac{f_i^\alpha + f_j^\alpha}{(f_i^{\alpha\theta} + f_j^{\alpha\theta})(f_i^{\alpha(1-\theta)} + f_j^{\alpha(1-\theta)})}T_{ij}\right].$$

Using Lemma 1.3 twice, we find

$$(1.8) \quad \|Z\|_q \leq C^2\|T\|_q = C^2\|f^{\alpha(1-\theta)}Y + Yf^{\alpha(1-\theta)}\|_q.$$

Now, since $S = (f^{\alpha\theta}Z + Zf^{\alpha\theta})/2$ and $\alpha\theta = \frac{1}{p} - \frac{1}{q}$, by (1.1) we have $C_q(x) \leq \|Z\|_q$ and (1.8) implies

$$C_q(x) \leq C^2(\delta(\varepsilon)\|Y\|_2 + w(\varepsilon)\|S\|_p),$$

and since the inf of $\|Y\|_2$ over all factorizations of the form $x_k = (f^{\frac{1}{r}}y_k + y_kf^{\frac{1}{r}})/2$ (or equivalently $S = (f^{\frac{1}{r}}Y + Yf^{\frac{1}{r}})/2$) is equal to $C_2(x)$ we find

$$(1.9) \quad C_q(x) \leq C^2(\delta(\varepsilon)C_2(x) + w(\varepsilon)\|S\|_p)$$

by Step 1 and 2, for some c

$$\|x\|_p \leq c(\delta(\varepsilon)\|x\|_p + w(\varepsilon)\|S\|_p)$$

and hence choosing ε small enough we again conclude

$$\|x\|_p \leq c'\|S\|_p.$$

The preceding arguments, up to (1.9), work just as well if we merely assume that f has finite spectrum. We now use this to complete the proof in the general semifinite case.

Let (y_k) and $f \in \mathcal{D}$ be such that $x_k = (f^\alpha y_k + y_k f^\alpha)/2$ and $(\sum \|y_k\|_2)^{\frac{1}{2}} < 2C_2(x)$. Recalling Step 2, we have

$$(\sum \|y_k\|_2)^{\frac{1}{2}} \leq 2C''\|x\|_p.$$

Fix $\varepsilon' > 0$. Let g be a density with finite spectrum such that $\|g - f\|_1 < \varepsilon'$. We can find such a g by approximating the spectral decomposition of f so that the spectral decomposition of g commutes with that of f . Then, for any $\beta > 0$, we have clearly a bound $\|g^\beta - f^\beta\|_{\frac{1}{\beta}} \leq o(\varepsilon')$, and hence, if we wish, we can find a density g such that we actually have

$$(1.10) \quad \|f^\alpha - g^\alpha\|_{\frac{1}{\alpha}} < \varepsilon'.$$

Let $x'_k = (g^\alpha y_k + y_k g^\alpha)/2$ and $S' = \sum \xi_k \otimes x'_k$. Note

$$(1.11) \quad C_2(x') \leq (\sum \|y_k\|_2)^{\frac{1}{2}} \leq 2C''\|x\|_p.$$

By the proof of (1.9) applied with g in place of f we find

$$C_q(x') \leq C^2(\delta(\varepsilon)C_2(x') + w(\varepsilon)\|S'\|_p).$$

By Step 1 we have

$$|||x' |||_p \leq C' C_q(x') \leq C' C^2(\delta(\varepsilon) C_2(x') + w(\varepsilon) \|S'\|_p)$$

and hence by (1.11)

$$(1.12) \quad |||x' |||_p \leq C' C^2(2C''\delta(\varepsilon) |||x|||_p + w(\varepsilon) \|S'\|_p).$$

But clearly by Hölder and (1.10), we have an estimate $\|x_k - x'_k\|_p \leq o(\varepsilon')$ and hence we have both $|||x - x' |||_p \leq o(\varepsilon')$ and $\|S - S'\|_p \leq o(\varepsilon')$. Thus, letting $\varepsilon' \rightarrow 0$, we deduce from (1.12) that

$$|||x|||_p \leq C' C^2(2C''\delta(\varepsilon) |||x|||_p + w(\varepsilon) \|S\|_p).$$

and we conclude as before that

$$|||x|||_p \leq c' \|S\|_p.$$

Remark 1.6. We give a sketch of a proof of (1.6) using an ultraproduct argument. Clearly by a 2×2 trick, we may assume that $x = x^*$. Assuming (1.6) does not hold gives some $\varepsilon > 0$, a sequence of elements $x_n \in L_2(M, \tau)$, $f_n \in \mathcal{D}$ with $\|x_n\|_2 = 1$, $x_n = x_n^*$ with $\|f_n^\alpha x_n + x_n f_n^\alpha\|_p \leq \frac{1}{n}$ but $\|f_n^{\alpha(1-\theta)} x_n + x_n f_n^{\alpha(1-\theta)}\|_q \geq \varepsilon$.

We use the theory of ultrapowers from [28]. In the latter, Theorem 3.6 explains that given a free ultrafilter \mathfrak{U} on \mathbb{N} , there is a general (type III) von Neumann algebra \mathcal{A} so that there are natural identifications $\prod_{\mathfrak{U}} L_p(M, \tau) = L_p(\mathcal{A})$ for $p > 0$. Of course, taking powers and products commutes with the ultrapower construction (see Theorems 3.6 and 5.1 in [28]).

Consider $x = (x_n) \in L_2(\mathcal{A})$, $f = (f_n) \in L_2(\mathcal{A})$. We have $x = x^*$ with $\|x\|_2 = 1$, $f \geq 0$ with $\|f\|_1 = 1$ and $\|f^\alpha x + x f^\alpha\|_p = 0$ but $\|f^{\alpha(1-\theta)} x + x f^{\alpha(1-\theta)}\|_q \geq \varepsilon$.

From the definition of L_p -spaces associated to a type III von Neumann algebra (see [27, 34]), f and x can be seen as τ -measurable operators associated to the core $\tilde{\mathcal{A}}$ of \mathcal{A} which is semifinite with trace τ . Recall that the τ -measurable operators $L_0(\tilde{\mathcal{A}}, \tau)$ form a topological $*$ -algebra, we have $f^\alpha x = -x f^\alpha$. Hence $f^{2\alpha} x = -f^\alpha x f^\alpha = x f^{2\alpha}$, as $x = x^*$ so that $f^{2\alpha}$ and x strongly commute (see Lemma 2.3 in [15]). Thus x commutes with any spectral projection of f^α . But spectral projections of f and f^t coincide for any $t > 0$, we get that x and f^t commute. We have $\|2x f^\alpha\|_p = 0$, but for any spectral projection $p = 1_{(a,b)}(f)$ with $0 < a < b < \infty$, there is some $v \in \tilde{\mathcal{A}}$ with $f^\alpha v = p f^{\alpha(1-\theta)}$, so $x f^{\alpha(1-\theta)} p = 0$. Letting $a \rightarrow 0$ and $b \rightarrow \infty$, $p f^{\alpha(1-\theta)}$ converges to $f^{\alpha(1-\theta)}$ in $L_0(\tilde{\mathcal{A}}, \tau)$, hence $x f^{\alpha(1-\theta)} = 0$. This contradicts $\|2x f^{\alpha(1-\theta)}\|_q \geq \varepsilon$.

The trace τ on M does not play any rôle in the above argument. Thus (1.6) holds for any type III von Neumann algebra M with $f \in L_1(M)^+$ and $\|f\|_1 = 1$.

2 The new case $0 < p < 1$

The proofs in this section are valid for $0 < p < 2$ but are really pertinent only for $0 < p < 1$. For simplicity, to avoid distinguishing the normed case from the p -normed one, we assume $0 < p < 1$ throughout. We will compensate for the lack of convexity with subharmonicity. Indeed, it is well known that on an L_p -space, commutative or not, the norm, as well as the function $x \mapsto \|x\|_p^p$, is subharmonic. We will use moreover certain inequalities which express its “uniform subharmonicity”, in analogy with the uniform convexity of L_p when $p > 1$.

Let $0 < p < s \leq \infty$. In this section, we set

$$\alpha = \frac{1}{r} = \frac{1}{p} - \frac{1}{s}.$$

The previous section corresponds to the particular value $s = 2$.

Let x be in $L_s(\tau)$, and let $f \in L_1^+$ with $\|f\|_1 = 1$. Note that $\|f^\alpha\|_r = 1$. Let $0 < \theta < 1$. Let q be determined by

$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{s}.$$

Our main result is a new form of non-commutative Hölder inequality:

Theorem 2.1. *Let $0 < p < q < s \leq \infty$. Let α, θ be as above. Then for any $0 < R < p$ there is a constant C such that for any $x \in L_s(\tau)$ and $f \in L_1(\tau)^+$ with $\|f\|_1 = 1$, and for any unitaries $V, W \in M$ commuting with f we have*

$$(2.1) \quad \|xWf^{\alpha(1-\theta)} + Vf^{\alpha(1-\theta)}x\|_q \leq C \|xWf^\alpha + Vf^\alpha x\|_p^{\frac{R}{2}(1-\theta)} \|x\|_s^{1-\frac{R}{2}(1-\theta)}.$$

In particular for any choice of sign ± 1 we have

$$(2.2) \quad \|xf^{\alpha(1-\theta)} \pm f^{\alpha(1-\theta)}x\|_q \leq C \|xf^\alpha \pm f^\alpha x\|_p^{\frac{R}{2}(1-\theta)} \|x\|_s^{1-\frac{R}{2}(1-\theta)}.$$

Since this implies that (1.6) holds with $\delta(\varepsilon) = O(\varepsilon^{\frac{R}{2}(1-\theta)})$ (with $s = 2$), by Remark 1.5 we have

Corollary 2.2. *The implication $K_q \Rightarrow K_p$ remains valid for any $0 < p < 1$. In particular the non-commutative Khintchine inequality (0.2) holds for any $0 < q < 1$.*

Corollary 2.3. *Let $0 < p(1), p(2) < \infty$ and let $u : L_{p(1)}(\tau) \rightarrow L_{p(2)}(\tau)$ be any bounded linear operator. Then there is a constant $C(p(1), p(2))$ such that for any finite sequence (x_j) in $L_{p(1)}(\tau)$ we have*

$$|||(u(x_j))|||_{p(2)} \leq C(p(1), p(2)) \|u\| |||(x_j)|||_{p(1)}.$$

Proof. Since this is clear when $|||(x_j)|||_{p(1)}$ and $|||(u(x_j))|||_{p(2)}$ are replaced by the corresponding Rademacher averages, this corollary follows from the Khintchine inequalities, now extended to the whole range $0 < p < \infty$. Note that it is well known that the Kahane inequalities remain valid for quasi-normed spaces. \square

We denote by $U = \{z \in \mathbb{C} \mid 0 < \Re(z) < 1\}$ the classical vertical strip of unit width of the complex plane.

Remark 2.4. It seems worthwhile to start by a rough outline of the proof of Theorem 2.1. The natural way to prove (2.1) (and we will use this in the end) is to introduce the analytic function $G_0(z) = xWf^{\alpha(1-z)} + Vf^{\alpha(1-z)}x$ defined for $z \in U$ and to use some form of the 3-line lemma, as in (2.4) below, to estimate $\|G_0(\theta)\|_q$. In order to do so, we need to have bounds on the two boundary vertical lines. The bound for $z = 1 + it$ of the form $\|G_0(1 + it)\|_s \leq c\|x\|_s$ is straightforward. The problem is the missing bound

$$\|G_0(it)\|_p \leq c\|G_0(0)\|_p \quad ?$$

which seems highly unrealistic. However, it turns out that using the complex uniform convexity (and the simple algebraic form of G_0) as in (2.7), we will be able to majorize, for any fixed $\gamma > 1$, the function

$$G(z) = G_0(\gamma z + 1 - \gamma) = xWf^{\gamma\alpha(1-z)} + Vf^{\gamma\alpha(1-z)}x,$$

so that we have for $0 < \omega < 1$ and $0 < R < p$ such that $\frac{1}{q} = \frac{1-\omega}{R} + \frac{\omega}{s}$

$$\left(\int \|G(it)\|_R^R Q_\omega^0(dt) \right)^{\frac{1}{R}} \lesssim \|x\|_s^{1-\frac{R}{2}} \|G_0(0)\|_p^{\frac{R}{2}}.$$

Again we have $\|G(1+it)\|_s \leq c\|x\|_s$. Denote $1-\omega = \frac{1-\theta}{\gamma}$ so that $G(\omega) = G_0(\theta)$. Thus, applying the 3-line type argument (see (2.4) below) to the function G , we obtain a bound of the form

$$\|G_0(\theta)\|_q = \|G(\omega)\|_q \lesssim \left(\|x\|_s^{1-\frac{R}{2}} \|G_0(0)\|_p^{\frac{R}{2}}\right)^{1-\omega} \|x\|_s^\omega$$

which yields (2.1).

We will use the well known fact that complex interpolation remains valid for the $L_p(\tau)$ spaces in the range $0 < p < 1$. We will need just one direction. (Note however that the argument given for this fact for Schatten classes at the end of [6] is erroneous.) For simplicity we restrict the discussion here to the finite case.

Lemma 2.5. *Assume given (M, τ) as before with τ finite. Let $0 < p_0 < p_1 \leq \infty$. Let G be a bounded analytic function on U with values in $L_{p_1}(\tau)$, admitting a.e. non-tangential boundary values. Let us set for $j = 0, 1$*

$$g(j+it) = \|G(j+it)\|_{L_{p_j}(\tau)}.$$

For any $0 < \theta < 1$, let p_θ be such that $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Let \mathbb{P}_U^θ denote the probability measure which is the harmonic measure associated to θ with respect to U . We have then

$$(2.3) \quad \log \|G(\theta)\|_{L_{p_\theta}(\tau)} \leq \int_{\partial U} \log g(\xi) \mathbb{P}_U^\theta(d\xi).$$

Let Q_θ^0 (resp. Q_θ^1) be the probability on $\{\xi \in \mathbb{C} \mid \Re(\xi) = 0\}$ (resp. $\{\xi \in \mathbb{C} \mid \Re(\xi) = 1\}$) such that $\mathbb{P}_U^\theta = (1-\theta)Q_\theta^0 + \theta Q_\theta^1$, we have then

$$(2.4) \quad \|G(\theta)\|_{L_{p_\theta}(\tau)} \leq \left(\int g(it)^{p_0} Q_\theta^0(dt)\right)^{\frac{1-\theta}{p_0}} \left(\int g(1+it)^{p_1} Q_\theta^1(dt)\right)^{\frac{\theta}{p_1}}.$$

Sketch. We freely use conformal equivalence with the disc to justify the technical points.

The inequality (2.3) is well known when $p_0 \geq 1$. We will show that if it holds for a pair p_0, p_1 then it also holds for the pair $\frac{p_0}{2}, \frac{p_1}{2}$. This clearly suffices to cover the whole range $0 < p_0 < p_1 \leq \infty$.

Fix $\varepsilon > 0$. Let

$$\forall \xi \in \partial U \quad w(\xi) = (G(\xi)^* G(\xi))^{\frac{1}{2}} + \varepsilon 1 = |G(\xi)| + \varepsilon 1.$$

Following a well established tradition, we claim that G admits a factorization as a product of analytic functions on U :

$$G = G_1 G_2$$

satisfying

$$G_1 G_1^* \leq |G^*| = (GG^*)^{\frac{1}{2}} \text{ and } G_2^* G_2 \leq w \text{ on } \partial U.$$

Indeed, by the classical operator valued analogue of Szegő's theorem (see e.g. Th. 8.1 in [27] and the Remark (i) after it), there is a bounded analytic function F with values in $L_{2p_1}(\tau)$ such that

$$\forall \xi \in \partial U \quad F(\xi)^* F(\xi) = w(\xi).$$

Moreover, since $w \geq \varepsilon 1$, the function $z \mapsto F(z)^{-1}$ is well defined and bounded on U . Let then $G_1 = GF^{-1}$ and $G_2 = F$. Let $G = u|G|$ be the polar decomposition. Then, as is classical, we have $GG^* = u|G|^2 u^*$ and $|G^*| = u|G| u^*$. Moreover, since $\lambda \mapsto \lambda^{\frac{1}{2}}(\lambda + \varepsilon)^{-1} \lambda^{\frac{1}{2}}$ is at most 1 on \mathbb{R}_+ , we clearly have $|G|^{\frac{1}{2}} w^{-1} |G|^{\frac{1}{2}} \leq 1$. Thus we have on ∂U

$$G_1 G_1^* = G(F^* F)^{-1} G^* = G w^{-1} G^* = u|G| w^{-1} |G| u^* \leq u|G| u^* = |G^*|.$$

This proves our claim.

Let $g_1(j+it) = \|G_1(j+it)\|_{L_{p_j}(\tau)}$ and $g_2(j+it) = \|G_2(j+it)\|_{L_{p_j}(\tau)}$. Assume the Lemma holds for the pair p_0, p_1 . Then G_1 and G_2 satisfy (2.3). Therefore since by Hölder

$$\|G(\theta)\|_{L_{p_\theta/2}(\tau)} \leq \|G_1(\theta)\|_{L_{p_\theta}(\tau)} \|G_2(\theta)\|_{L_{p_\theta}(\tau)}$$

and also $\|G_1(j+it)\|_{L_{p_j}(\tau)}^2 = \|G(j+it)\|_{L_{\frac{p_j}{2}}(\tau)}^2$ and $\|G_2(j+it)\|_{L_{p_j}(\tau)}^2 = \|w(j+it)\|_{L_{\frac{p_j}{2}}(\tau)}^2$, if we add the two inequalities (2.3) written for G_1 and G_2 we obtain

$$\begin{aligned} 2 \log \|G(\theta)\|_{L_{p_\theta/2}(\tau)} &\leq (1-\theta) \int \log \|G(it)\|_{L_{p_0/2}(\tau)} Q_\theta^0(dt) + \theta \int \log \|G(1+it)\|_{L_{p_1/2}(\tau)} Q_\theta^1(dt) \\ &+ (1-\theta) \int \log \|w(it)\|_{L_{p_0/2}(\tau)} Q_\theta^0(dt) + \theta \int \log \|w(1+it)\|_{L_{p_1/2}(\tau)} Q_\theta^1(dt). \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we obtain (2.3), and (2.4) follows, as in the classical case, using the convexity of the exponential function. \square

We will use a certain form of uniform convexity inequalities for Hardy spaces with values in non-commutative L_p -spaces for $0 < p \leq 1$, extending the case $p = 1$ which is treated in [10]. See [7] for more on complex uniform convexity (and in particular Haagerup's inequality included as Th. 4.3 in [7]). We refer the interested reader to [35] for early estimates of the moduli of uniform convexity of the Schatten classes S_p for $1 < p \leq 2$. See also [2] and more recently [31] for optimal constants in the associated martingale inequalities. The next result appears as Th. 3.1 in [37].

Theorem 2.6 (Q. Xu). *Let $0 < p \leq 2$. There is a constant $\delta_p > 0$ such that for any function $F \in H_p(D; L_p(\tau))$ we have*

$$(2.5) \quad \|F(0)\|_{L_p(\tau)}^2 + \delta_p \|F - F(0)\|_{H_p(D; L_p(\tau))}^2 \leq \|F\|_{H_p(D; L_p(\tau))}^2.$$

Remark 2.7. A similar inequality holds for the values $F(\zeta)$ of F at another point ζ of D . Indeed, using a Möbius map as conformal equivalence taking 0 to ζ , we find

$$(2.6) \quad \|F(\zeta)\|_{L_p(\tau)}^2 + \delta_p \|F - F(\zeta)\|_{L_p(\mu^\zeta; L_p(\tau))}^2 \leq \|F\|_{L_p(\mu^\zeta; L_p(\tau))}^2,$$

where μ^ζ denotes the Poisson probability (harmonic) measure on ∂D associated to $\zeta \in D$.

Remark 2.8. By conformal equivalence, a similar inequality holds with the unit strip

$$U = \{z \in \mathbb{C} \mid 0 < \Re(z) < 1\}$$

in place of the unit disc D . For any $0 < \theta < 1$, recall that \mathbb{P}_U^θ denotes the probability measure which is the Poisson or harmonic measure associated to θ with respect to the strip U . Then (2.6) becomes

$$(2.7) \quad \|F(\theta)\|_{L_p(\tau)}^2 + \delta_p \|F - F(\theta)\|_{L_p(\mathbb{P}_U^\theta; L_p(\tau))}^2 \leq \|F\|_{L_p(\mathbb{P}_U^\theta; L_p(\tau))}^2.$$

Proof of Theorem 2.1. First we reduce the situation where $x = x^*$, indeed let us assume the result hold in this situation and consider the 2×2 -matrices

$$(2.8) \quad \tilde{W} = \begin{bmatrix} V^* & 0 \\ 0 & W \end{bmatrix}, \tilde{V} = \begin{bmatrix} V & 0 \\ 0 & W^* \end{bmatrix}, \tilde{f} = \frac{1}{2} \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}, \tilde{x} = \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix}.$$

Those elements satisfy the assumptions in $M_2(M)$ and \tilde{x} is selfadjoint. But one has

$$\|\tilde{x}\tilde{W}\tilde{f}^{\alpha(1-\theta)} + \tilde{V}\tilde{f}^{\alpha(1-\theta)}\tilde{x}\|_q = 2^{\frac{1}{q}-1}\|xWf^{\alpha(1-\theta)} + Vf^{\alpha(1-\theta)}x\|_q,$$

$$\|\tilde{x}\tilde{W}\tilde{f}^\alpha + \tilde{V}\tilde{f}^\alpha\tilde{x}\|_p = 2^{\frac{1}{p}-1}\|xWf^\alpha + Vf^\alpha x\|_p.$$

So that (2.1) for x, f, V, W follows from that of $\tilde{x}, \tilde{f}, \tilde{V}, \tilde{W}$.

Next we reduce the proof to finite von Neumann algebras. To see that, assume the result is true for finite von Neumann algebras. Let $p_n = 1_{(\frac{1}{n}, \infty)}(f)$. Note that p_n commutes with f and V . We have $p_n \rightarrow p_\infty = 1_{(0, \infty)}(f)$ in M for the strong operator topology. This implies that for any $t < \infty$ and $y \in L_t(\tau)$ $\|p_n y p_n - p_\infty y p_\infty\|_t \rightarrow 0$ (and $\|p_n y p_n\|_\infty \leq \|y\|_\infty$ if $y \in M$). As $p_n M p_n$ is finite, we can apply the result to $p_n x p_n, f p_n$ and $V p_n$ and let $n \rightarrow \infty$. Note that $f(1 - p_\infty) = (1 - p_\infty)f = 0$. There is a remaining term of the form $(1 - p_\infty)xWf^{\alpha(1-\theta)}p_\infty + p_\infty Vf^{\alpha(1-\theta)}x(1 - p_\infty)$ which is easy to handle, it splits in two terms that can be treated using basic one sided estimates.

Next we reduce the proof to the technically easier case when f has a finite spectrum. Fix $\varepsilon' > 0$. Let $g \in L_1^+$ with finite spectrum such that $\|g - f\|_1 < \varepsilon'$. We can find such a g by approximating the spectral decomposition of f so that spectral decomposition of g commutes with f . Then, for any $\beta > 0$, we have clearly a bound $\|g^\beta - f^\beta\|_{\frac{1}{\beta}} \leq o(\varepsilon')$, from which it follows, by Hölder, that for any $y \in L_s(\tau)$ we have $\|y(g^\beta - f^\beta)W + V(g^\beta - f^\beta)y\|_{s+\frac{1}{\beta}} \leq o(\varepsilon')$. A fortiori, we have

$$\left| \|yg^\beta W + Vg^\beta y\|_{s+\frac{1}{\beta}} - \|yWf^\beta + Vf^\beta y\|_{s+\frac{1}{\beta}} \right| \leq o(\varepsilon).$$

From this last inequality it becomes clear that we may reduce the proof of (2.1) to the case when f has a finite spectrum and $x = x^*$, so we assume this in the rest of the proof.

Fix $1 < \gamma$. We will apply (2.7) to the analytic function $F : U \rightarrow L_R(\tau)$ defined by

$$F(\zeta) = f^{\gamma\alpha\zeta} x f^{\gamma\alpha(1-\zeta)},$$

where $\frac{1}{R} = \frac{1}{s} + \alpha\gamma$ and we replace θ in (2.7) by $\frac{1}{\gamma}$. We have

$$F\left(\frac{1}{\gamma}\right) = f^\alpha x f^{\alpha(\gamma-1)}.$$

Note that $\frac{1}{R} = \frac{1}{p} + \alpha(\gamma-1)$ so that by Hölder, multiplication by $f^{\alpha(\gamma-1)}$ (left or right) is of norm 1 from $L_p(\tau)$ to $L_R(\tau)$. As $x = x^*$, the right hand side of (2.7) is exactly $\|x f^{\alpha\gamma}\|_R^2 = \|f^{\alpha\gamma} x\|_R^2$. The R -triangle inequality gives (note that $R < p \leq 1$):

$$\|x f^{\alpha\gamma}\|_R^R = \|x f^{\alpha\gamma} W\|_R^R \leq \|F\left(\frac{1}{\gamma}\right)\|_R^R + \|(xWf^\alpha + Vf^\alpha x)f^{\alpha(\gamma-1)}\|_R^R \leq \|F\left(\frac{1}{\gamma}\right)\|_R^R + \|xWf^\alpha + Vf^\alpha x\|_p^R.$$

Let us assume for the moment that $\|xWf^\alpha + Vf^\alpha x\|_p \leq \|x f^{\alpha\gamma}\|_R$, so that by convexity of $t \mapsto t^{\frac{2}{R}}$:

$$\|F\left(\frac{1}{\gamma}\right)\|_R^2 \geq \|x f^{\alpha\gamma}\|_R^2 - \frac{2}{R} \|x f^{\alpha\gamma}\|_R^{2-R} \|xWf^\alpha + Vf^\alpha x\|_p^R.$$

Thus, we get

$$\|F\|_{L_R(\mathbb{P}_U^{\frac{1}{\gamma}}; L_R(\tau))}^2 - \|F\left(\frac{1}{\gamma}\right)\|_{L_R(\tau)}^2 \leq \frac{2}{R} \|x f^{\alpha\gamma}\|_R^{2-R} \|xWf^\alpha + Vf^\alpha x\|_p^R.$$

Let $u_t = f^{\gamma\alpha}it$. By (2.7), we have (a fortiori)

$$\delta_R \left(\frac{1}{\gamma} \int \|f^\alpha x f^{\alpha(\gamma-1)} - u_t f^{\gamma\alpha} x u_t^*\|_R^R Q_{\frac{1}{\gamma}}^1(dt) \right)^{\frac{2}{R}} \leq \frac{2}{R} \|x f^{\alpha\gamma}\|_R^{2-R} \|x W f^\alpha + V f^\alpha x\|_p^R.$$

Put $H(t) = u_t f^{\gamma\alpha} x - f^\alpha x f^{\alpha(\gamma-1)} u_t$, so that under the assumption $\|x W f^\alpha + V f^\alpha x\|_p \leq \|x f^{\alpha\gamma}\|_R$:

$$(2.9) \quad \int \|H(t)\|_R^R Q_{\frac{1}{\gamma}}^1(dt) \leq C_R \|x W f^\alpha + V f^\alpha x\|_p^{\frac{R^2}{2}} \|x\|_s^{R-\frac{R^2}{2}}.$$

We now introduce the analytic function $G : U \rightarrow L_R(\tau)$ defined by

$$G(z) = V f^{\gamma\alpha(1-z)} x + x W f^{\gamma\alpha(1-z)}.$$

We apply (2.4) at the point $\omega = \frac{\gamma+\theta-1}{\gamma} \in U$ with $p_1 = s$ and $p_0 = R$. Easy computations give

$$1 - \omega = \frac{1 - \theta}{\gamma}, \quad \frac{1}{p_\omega} = \frac{1 - \omega}{R} + \frac{\omega}{s} = \frac{1}{q}.$$

Note that for any $t \in \mathbb{R}$ we have (recall $\chi_p = \max\{2^{\frac{1}{p}-1}, 1\}$)

$$\|G(1 + it)\|_s \leq 2\chi_s \|x\|_s.$$

For $t \in \mathbb{R}$, we also have

$$(2.10) \quad G(it) = V u_{-t} f^{\gamma\alpha} x + x f^{\gamma\alpha} W u_{-t} = V H(-t) + (V f^\alpha x + x f^\alpha W) f^{\alpha(\gamma-1)} u_{-t}.$$

We want to estimate $\int \|G(it)\|_R^R Q_\omega^0(dt)$ in order to be able to apply (2.4).

We will distinguish between two cases: either $\|x W f^\alpha + V f^\alpha x\|_p$ is larger than $\|x f^{\alpha\gamma}\|_R$ or not.

If $\|x W f^\alpha + V f^\alpha x\|_p > \|x f^{\alpha\gamma}\|_R = \|f^{\alpha\gamma} x\|_R$, then clearly (by the R -triangle inequality)

$$(2.11) \quad \int \|G(it)\|_R^R Q_\omega^0(dt) \leq 2 \|x W f^\alpha + V f^\alpha x\|_p^R.$$

Otherwise with the Hölder inequality, the R -triangle inequality and (2.10):

$$\int \|G(it)\|_R^R Q_\omega^0(dt) \leq \int \|H(-t)\|_R^R Q_\omega^0(dt) + \|V f^\alpha x + x f^\alpha W\|_p^R.$$

But, by the explicit formula for the Poisson kernels on the strip (see [5] page 93), namely

$$Q_\omega^k(t) = \frac{\sin \pi \omega}{2(\cosh(\pi t) - (-1)^k \cos(\pi \omega))}, \quad (k = 0, 1, 0 < \omega < 1)$$

we know that $Q_{\frac{1}{\gamma}}^1(t)dt$ and $Q_\omega^0(t)dt$ are equivalent symmetric measures.

Thus, in the second case, by (2.9) we can find some constant M_γ with

$$\int \|G(it)\|_R^R Q_\omega^0(dt) \leq M_\gamma \|x W f^\alpha + V f^\alpha x\|_p^{\frac{R^2}{2}} \|x\|_s^{R-\frac{R^2}{2}}.$$

Since $\|x W f^\alpha + V f^\alpha x\|_p \leq \|x\|_s$, by (2.11) a similar bound also holds in the first case (and hence in both cases). By (2.4) there is a constant C_γ depending on γ (and p, q and s) so that:

$$\begin{aligned} \|V f^{\alpha(1-\theta)} x + x W f^{\alpha(1-\theta)}\|_q &\leq C_\gamma \|x W f^\alpha + V f^\alpha x\|_p^{\frac{R(1-\omega)}{2}} \|x\|_s^{(1-\frac{R}{2})(1-\omega)+\omega} \\ &= C_\gamma \|x W f^\alpha + V f^\alpha x\|_p^{\frac{R(1-\theta)}{2\gamma}} \|x\|_s^{1-\frac{R(1-\theta)}{2\gamma}}. \end{aligned}$$

The theorem is obtained by letting $\gamma \rightarrow 1$ (note that $C_\gamma \rightarrow \infty$). □

Remark 2.9. In Theorem 2.1 with the same notation, one can also get for $x \in L_s(\tau)$ with $f, g \in L_1(\tau)^+$ of norm 1 and V a unitary commuting with f and W a unitary commuting with g

$$\|xg^{\alpha(1-\theta)}W + Vf^{\alpha(1-\theta)}x\|_q \leq C\|xg^\alpha W + Vf^\alpha x\|_p^{\frac{R}{2}(1-\theta)}\|x\|_s^{1-\frac{R}{2}(1-\theta)}.$$

To see it, one can use the 2×2 trick of (2.8) but with $\tilde{f} = \frac{1}{2} \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}$.

Remark 2.10. One can now deduce an estimate weaker than (1.4), with the notation of Theorem 2.1:

$$\max \{ \|f^\alpha x\|_p, \|xf^\alpha\|_p \} \lesssim \|f^\alpha x + xf^\alpha\|_p^{\frac{R}{4}} \|x\|_s^{1-\frac{R}{4}}.$$

This follows from the identity $2f^\alpha x = f^\alpha x + xf^\alpha + f^{\frac{\alpha}{2}}(f^{\frac{\alpha}{2}}x + xf^{\frac{\alpha}{2}}) - (f^{\frac{\alpha}{2}}x + xf^{\frac{\alpha}{2}})f^{\frac{\alpha}{2}}$ and Theorem 2.1 with $\theta = \frac{1}{2}$.

3 Maurey's factorization of operators with values in L_p , $0 < p < 1$

In this section we attempt to give a non-commutative version of the following result of Maurey [21]: Let (Ω, μ) be any measure space and let $0 < p < 1$. For any bounded linear operator $u : H \rightarrow L_p(\mu)$ with $\|u\| \leq 1$ there is a probability density f such that (with the convention $\frac{0}{0} = 0$)

$$\forall x \in H \quad \left(\int \left| \frac{u(x)}{f^\alpha} \right|^2 d\mu \right)^{\frac{1}{2}} \leq C\|x\|,$$

where C is a constant depending only on p and where, as before, $\alpha = \frac{1}{p} - \frac{1}{2}$. We refer to [17, 19, 14] for results related to this section.

We will use the following well known variant of the Hahn-Banach theorem (see e.g. [24, p.39 and p.421]).

Lemma 3.1. *Let S be a set and let $\mathcal{F} \subset \ell_\infty(S)$ be a convex cone of real valued functions on S such that*

$$\forall f \in \mathcal{F} \quad \sup_{s \in S} f(s) \geq 0.$$

Then there is a net (λ_i) of finitely supported probability measures on S such that

$$\forall f \in \mathcal{F} \quad \lim_{\mathcal{U}} \int f d\lambda_i \geq 0$$

for any ultrafilter \mathcal{U} refining the net.

It will be convenient to use the following notation:

Let f be any element in $(M_*)_+$, and as before $\alpha = \frac{1}{p} - \frac{1}{2}$. We denote by $J(f)$ (J for Jordan) the mapping $x \mapsto \frac{fx+xf}{2}$. We will denote by $L(f)$ and $R(f)$ the left and right multiplications by f , so that $J(f) = \frac{L(f)+R(f)}{2}$. We use the same notation for

$$J(f^\alpha)(x) = \frac{f^\alpha x + x f^\alpha}{2}$$

for any $\alpha > 0$. Note that since $f^\alpha \in L_{\frac{1}{\alpha}}(\tau)$ the latter mapping is bounded from $L_2(\tau)$ to $L_p(\tau)$ when $\alpha = \frac{1}{p} - \frac{1}{2}$. Moreover, this mapping preserves self-adjointness.

In addition, if f has full support (*i.e.* if f is a faithful normal state on M) then the mapping $J(f^\alpha) : L_2(\tau) \rightarrow L_p(\tau)$ is injective. We essentially already observed this (see Remark 1.6). Indeed, if $J(f^\alpha)(x) = 0$ for $x \in L_2(\tau)$ then the same is true for the real and imaginary parts of x , so we may assume $x = x^*$. Then $J(f^\alpha)(x) = 0$ implies $f^{2\alpha}x = xf^{2\alpha}$ and hence, since x is self-adjoint, $f^\alpha x = xf^\alpha$. But then $J(f^\alpha)(x) = f^\alpha x$, and since f has full support, $f^\alpha x = 0$ implies $x = 0$.

In this section, we will assume that M is σ -finite. Then (see [32, p. 78]) there is a faithful normal state f_0 on M . It will be convenient to invoke the following elementary Lemma.

Lemma 3.2. *Let $f \in \mathcal{D}$ and let $g = (f^{2\alpha} + f_0^{2\alpha})^{\frac{1}{2\alpha}}$. Then g is faithful and $\tau(g) \leq 2$. Moreover*

$$J(f^\alpha)(L_2(\tau)) \subset J(g^\alpha)(L_2(\tau))$$

and for any $y \in L_2(\tau)$ we have

$$(3.1) \quad \|J(g^\alpha)^{-1}J(f^\alpha)y\|_2 \leq 2\|y\|_2.$$

Proof. Since $g \geq f_0$, g is faithful. Let $s = \frac{1}{2\alpha}$. Note $s \leq 1$. By the s -triangle inequality (see (0.3)), we have $\tau(g) \leq \tau(f) + \tau(f_0) \leq 2$. We will prove more generally that

$$L(f^\alpha)(L_2(\tau)) + R(f^\alpha)(L_2(\tau)) \subset J(g^\alpha)(L_2(\tau)).$$

Note that $f^{2\alpha} \leq g^{2\alpha}$. Therefore, as unbounded operators on $L_2(\tau)$ we have

$$L(f^{2\alpha}) = (L(f^\alpha))^2 \leq (L(f^\alpha))^2 + 2L(f^\alpha)R(f^\alpha) + (R(f^\alpha))^2 = 4(J(g^\alpha))^2.$$

This implies

$$\|(J(g^\alpha))^{-1}L(f^\alpha)\|_{L_2(\tau) \rightarrow L_2(\tau)} \leq 2.$$

Thus if $x = f^\alpha y$ with $y \in L_2(\tau)$, we have $x = J(g^\alpha)(y')$ with $y' = (J(g^\alpha))^{-1}L(f^\alpha)(y) \in L_2(\tau)$ and $\|y'\|_2 \leq 2\|y\|_2$. A similar result holds for the right hand side multiplication. This proves the announced inclusion and (3.1) follows by the triangle inequality in $L_2(\tau)$. \square

We will denote by \mathcal{D}' the subset of \mathcal{D} formed of the elements with full support. Assuming $f \in \mathcal{D}'$ and $T \in L_p(\tau)$, we denote

$$D_{f^\alpha}(T) = \|(J(f^\alpha))^{-1}(T)\|_2 \quad \text{if } T \in J(f^\alpha)(L_2(\tau))$$

and we set $D_{f^\alpha}(T) = \infty$ if T is not in this range.

Theorem 3.3. *Let $0 < p < 1$. There is a constant C for which the following holds: Let $u : H \rightarrow L_p(\tau)$ be a bounded operator with $\|u\| \leq 1$. Then there is a net of finitely supported probability measures (λ_i) on \mathcal{D}' such that for any $x \in H$ we have*

$$(3.2) \quad \left(\lim_{\mathcal{U}} \int [D_{f^\alpha}(u(x))]^2 d\lambda_i(f) \right)^{\frac{1}{2}} \leq C\|x\|_H,$$

where \mathcal{U} is any ultrafilter refining the net.

Proof of Theorem 3.3. For any finite sequence (x_k) in H we have by (0.2)

$$\begin{aligned} |||(u(x_k))|||_p &\leq \beta_p \left(\int \left\| \sum r_k(t)u(x_k) \right\|_p^p dt \right)^{\frac{1}{p}} \\ &\leq \beta_p \left(\int \left\| \sum r_k(t)x_k \right\|_H^2 dt \right)^{\frac{1}{2}} = \beta_p \left(\sum \|x_k\|_H^2 \right)^{\frac{1}{2}}. \end{aligned}$$

By Step 2 from §1 and by Lemma 3.2 it follows that for some constant β'_p we have

$$\inf_{f \in \mathcal{D}'} \sum D_{f^\alpha}(u(x_k))^2 \leq \beta_p'^2 \sum \|x_k\|_H^2.$$

Let $a_k > 0$ be arbitrary coefficients. We may obviously replace x_k by $a_k x_k$. Let us now fix a sequence (T_k) in $L_p(\tau)$. Assume given $\beta > 0$ such that for any sequence $a_k > 0$ we have

$$(3.3) \quad \inf_{f \in \mathcal{D}'} \sum a_k^2 D_{f^\alpha}(T_k)^2 \leq \beta \sum a_k^2.$$

Note that this obviously remains true if we replace the infimum over \mathcal{D}' by an infimum over the set \mathcal{D}_T formed of those $f \in \mathcal{D}'$ such that $D_{f^\alpha}(T_k) < \infty$ for all $k = 1, \dots, n$. By Lemma 3.1 there is a net of finitely supported probability measures (λ_i) on \mathcal{D}_T and an ultrafilter \mathcal{U} such that for any $k = 1, \dots, n$ we have

$$\lim_{\mathcal{U}} \int [D_{f^\alpha}(T_k)]^2 \lambda_i(df) \leq \beta.$$

Therefore, since (3.3) holds for $T_k = u(x_k)$ and $\beta = \beta_p'^2$, for any finite sequence $x_k \in B_H$ there is a net of finitely supported probability measures (λ_i) on \mathcal{D}' such that for any $k = 1, \dots, n$

$$\lim_{\mathcal{U}} \int [D_{f^\alpha}(u(x_k))]^2 \lambda_i(df) \leq \beta_p'^2.$$

Then after a simple rearrangement of this net (e.g. by indexing our net by the set of finite subsets of B_H) we obtain a net (λ_i) of finitely supported probability measures on \mathcal{D}' such that

$$\forall x \in H \quad \lim_{\mathcal{U}} \int [D_{f^\alpha}(u(x))]^2 \lambda_i(df) \leq \beta_p'^2 \|x\|_H^2.$$

□

Remark 3.4. When $f \geq 0$ is bounded, the multiplications $L(f)$ and $R(f)$ are self-adjoint nonnegative operators on $L_2(\tau)$ and hence the same is true for $J(f)$. Moreover for any $0 \leq g \leq f$ we have $J(g) \leq J(f)$ and hence $J(f)^{-1} \leq J(g)^{-1}$. Now if $p \geq 1$ since $2\alpha \leq 1$ it follows that $x \mapsto x^{-2\alpha}$ is operator convex on \mathbb{R}_+ (see e.g. [3]), so we can deduce from (3.2) that there is a net of densities $f_i \in \mathcal{D}'$ (namely $f_i = \int f d\lambda_i(f)$) such that for some constant C for any $x \in H$

$$\lim_{\mathcal{U}} [D_{f_i^\alpha}(u(x))]^2 \leq C \|x\|_H^2.$$

Then at least in the finite dimensional case, the net converges and we can find a density $g \in \mathcal{D}'$ such that we have, as in Maurey's original theorem,

$$[D_{g^\alpha}(u(x))]^2 \leq C \|x\|_H^2.$$

In the commutative case, the map $x \mapsto x^{-2\alpha}$ being convex on \mathbb{R}_+ for any α , this argument works also for $p < 1$. However, since $x \mapsto x^{-2\alpha}$ is not operator convex on \mathbb{R}_+ when $\alpha > \frac{1}{2}$, we do not see how to complete that same argument for $p < 1$.

4 Application to the Mazur maps

For a semifinite von Neumann algebra (M, τ) , the Mazur map $M_{p,q} : L_p(\tau) \rightarrow L_q(\tau)$ is given by $M_{p,q}(f) = f|f|^{\frac{p-q}{q}}$. It is known to be a uniform homeomorphism on spheres (see [28]) for $0 < p, q < \infty$. We would like to know for which exponents $0 < \gamma \leq 1$ this (non-linear) map is γ -Hölder, by which we mean that there is a constant C such that for any g, h in the unit sphere of $L_p(\tau)$ we have

$$\|M_{p,q}(g) - M_{p,q}(h)\|_q \leq C\|g - h\|_p^\gamma.$$

Precise estimates are given in [29] in the case $1 \leq p, q < \infty$, $M_{p,q}$ is Hölder with exponent $\min\{1, \frac{p}{q}\}$ as for commutative integration. Actually, it is shown there that for $0 < p, q < \infty$, $M_{p,q}$ is Hölder on all semifinite von Neumann algebras with exponent γ and constant C (both independent of M) iff the following inequalities occur for any finite von Neumann algebra M

$$(4.1) \quad \forall x \in M, x = x^*, \|x\|_\infty = 1, \forall f \in L_p(\tau)^+, \|f\|_p = 1, \quad \|xf^{\frac{p}{q}} \pm f^{\frac{p}{q}}x\|_q \leq C'\|xf \pm fx\|_p^\gamma.$$

Thus we can use Theorem 2.1 to get

Theorem 4.1. *For any $0 < p, q < \infty$ and any semifinite von Neumann algebra (M, τ) , the Mazur map $M_{p,q}$ is γ -Hölder for $\gamma < \frac{1}{2q}(\frac{p}{3^k})^2$ where $k \geq 0$ is the smallest integer such that $\frac{p}{q} < 3^k$.*

Proof. Using composition and [29], it suffices to do it when $0 < p, q \leq 1$.

We start by looking at $0 < p < q \leq 1$. By Theorem 2.1 with $s = \infty$, we have for all $R < p$ and some constant C_R , for x and f as in (4.1)

$$(4.2) \quad \|xf^{\frac{p}{q}} \pm f^{\frac{p}{q}}x\|_q \leq C_R\|xf \pm fx\|_p^{\frac{pR}{2q}}.$$

Thus (4.1) holds for any $\gamma < \frac{p^2}{2q}$, which gives us the case $k = 0$.

To treat the case $q < p \leq 1$, note that for any $g, h \in L_p$ with norm 1, using

$$\|g^*g - f^*f\|_{\frac{p}{2}} = \|g^*(g - f) + (g^* - f^*)f\|_{\frac{p}{2}} \leq 2^{\frac{2}{p}-1} \left(\|g^*(g - f)\|_{\frac{p}{2}} + \|(g^* - f^*)f\|_{\frac{p}{2}} \right)$$

and Hölder's inequality, we find

$$\||g|^2 - |h|^2\|_{\frac{p}{2}} \leq 2^{\frac{2}{p}}\|g - h\|_p.$$

One easily deduces that $M_{p,\frac{p}{3}}$ is 1-Hölder. By iteration, the same is true for $M_{p,\frac{p}{3^2}} = M_{\frac{p}{3},\frac{p}{3^2}}M_{p,\frac{p}{3}}$ and more generally for $M_{p,\frac{p}{3^k}}$ for all $k \geq 0$. Let k so that $\frac{p}{3^k} < q$. As $M_{p,q} = M_{\frac{p}{3^k},q}M_{p,\frac{p}{3^k}}$, one deduces that $M_{p,q}$ admits the same Hölder exponent as $M_{\frac{p}{3^k},q}$, which by (4.2) is at least as good as $\gamma_k < \frac{1}{2q}(\frac{p}{3^k})^2$. Thus one concludes that $M_{p,q}$ is γ_k -Hölder. \square

Remark 4.2. It seems likely that the exponents are not optimal. For instance, one can argue as in Corollary 2.4 in [29], using Kosaki's inequality $\|g^p - h^p\|_1 \leq \|g - h\|_p^p$ for $0 < p < 1$ and $g, h \in L_p^+$ to get that

$$\|xf^p - f^p x\|_1 \leq 2\|xf - fx\|_p^p.$$

Remark 4.3. Because of the lack of convexity and in particular of conditional expectations, one apparently cannot apply the Haagerup reduction technique to get the result for general von Neumann algebras.

References

- [1] J. Arazy and Y. Friedman, Contractive projections in C_p . *Mem. Amer. Math. Soc.* **95** (1992), no. 459, vi+109 pp.
- [2] K. Ball, E. Carlen, and E.H. Lieb, Sharp uniform convexity and smoothness inequalities for trace norms, *Invent. Math.* **115** (1994), 463–482.
- [3] R. Bhatia, Matrix analysis. Springer-Verlag, New York, 1997.
- [4] A. Buchholz, Optimal Constants in Khintchine Type Inequalities for Fermions, Rademachers and q-Gaussian Operators. *Bull Polish. Acad. N.* Vol. 53, No. 3, (2005) 315–321.
- [5] J. Bergh and J. Löfström, Interpolation spaces. An introduction. Springer Verlag, New York 1976.
- [6] F. Cobos, J. Peetre, and L.E. Persson, On the connection between real and complex interpolation of quasi-Banach spaces. *Bull. Sci. Math.* 122 (1998), 17–37.
- [7] W. J. Davis, D.J.H. Garling and N. Tomczak-Jaegermann, The complex convexity of quasi-normed linear spaces. *J. Funct. Anal.* **55** (1984), 110–150.
- [8] S. Dirksen and É. Ricard, Some remarks on noncommutative Khintchine inequalities. *Bull. Lond. Math. Soc.* 45 (2013), no. 3, 618–624.
- [9] U. Haagerup and M. Musat, On the best constants in noncommutative Khintchine-type inequalities, *J. Funct. Anal.* **250** (2007), 588–624.
- [10] U. Haagerup and G. Pisier, Factorization of analytic functions with values in non-commutative L_1 -spaces. *Canadian Journal of Math.* 41 (1989) 882–906.
- [11] A. Harcharras, Fourier analysis, Schur multipliers on S^p and non-commutative $\Lambda(p)$ -sets, *Studia Math.* **137** (1999), no. 3, 203–260.
- [12] F. Hiai and H. Kosaki, Means for matrices and comparison of their norms. *Indiana Univ. Math. J.* 48 (1999), no. 3, 899–936.
- [13] F. Hiai and H. Kosaki, Means of Hilbert space operators. Lecture Notes in Mathematics, 1820. Springer-Verlag, Berlin, 2003.
- [14] M. Junge and J. Parcet, Rosenthal’s theorem for subspaces of noncommutative L_p , *Duke Math. J.* **141** (2008), no. 1, 75–122.
- [15] H. Kosaki, Applications of uniform convexity of noncommutative L^p -spaces. *Trans. Amer. Math. Soc.* 283 (1984), no.1, 265–282.
- [16] F. Lust-Piquard, Inégalités de Khintchine dans C_p ($1 < p < \infty$), *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986), no. 7, 289–292.
- [17] F. Lust-Piquard, A Grothendieck factorization theorem on 2-convex Schatten spaces, *Israel J. Math.* **79** (1992), no. 2-3, 331–365.
- [18] F. Lust-Piquard and G. Pisier, Noncommutative Khintchine and Paley inequalities, *Ark. Mat.* **29** (1991), no. 2, 241–260.

- [19] F. Lust-Piquard and Q. Xu, The little Grothendieck theorem and Khintchine inequalities for symmetric spaces of measurable operators, *J. Funct. Anal.* **244** (2007), no. 2, 488–503.
- [20] B. Maurey, Une nouvelle démonstration d’un théorème de Grothendieck. Séminaire Maurey-Schwartz Année 1972–1973: Espaces L^p et applications radonifiantes, Exp. No. 22, 7 pp. Centre de Math., École Polytech., Paris, 1973.
- [21] B. Maurey, Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p . *Astérisque*, No. 11. Société Mathématique de France, Paris, 1974 ii+163 pp.
- [22] E. Nelson, Notes on non-commutative integration. *J. Functional Analysis* 15 (1974) 103–116.
- [23] G. Pisier, Grothendieck’s theorem for noncommutative C^* -algebras, with an appendix on Grothendieck’s constants, *J. Funct. Anal.* **29** (1978), no. 3, 397–415.
- [24] G. Pisier, Introduction to operator space theory, *London Mathematical Society Lecture Note Series*, 294, Cambridge University Press, Cambridge, 2003. viii+478 pp.
- [25] G. Pisier, Remarks on the non-commutative Khintchine inequalities for $0 < p < 2$. *J. Funct. Anal.* 256 (2009), 4128–4161.
- [26] G. Pisier, Grothendieck’s Theorem, past and present, *Bull. Amer. Math. Soc.* 49 (2012), 237–323.
- [27] G. Pisier and Q. Xu, Non-commutative L^p -spaces, *Handbook of the geometry of Banach spaces*, Vol. 2, 1459–1517, North-Holland, Amsterdam, 2003.
- [28] Y. Raynaud, On ultrapowers of non commutative L_p spaces. *J. Operator Theory*, 48(1):41–68, 2002.
- [29] E. Ricard, Hölder estimates for the noncommutative Mazur maps. *Arch. Math. (Basel)*, 104 (2015), no. 1, 37–45.
- [30] E. Ricard and Q. Xu, Complex interpolation of weighted non-commutative L_p -spaces. *Houston J. Math.* 37 (2011), 1165–1179.
- [31] E. Ricard and Q. Xu, A noncommutative martingale convexity inequality, *Ann. Probab.* To appear.
- [32] M. Takesaki, Theory of operator algebras. I. Springer-Verlag, Berlin, 1979.
- [33] M. Takesaki, Theory of operator algebras. II. Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003.
- [34] M. Terp, L^p spaces associated with von Neumann algebras. *Notes, Københavns Universitets Matematiske Institut*, 1981.
- [35] N. Tomczak-Jaegermann, The moduli of smoothness and convexity and the Rademacher averages of trace classes $S_p(1 \leq p < \infty)$. *Studia Math.* **50** (1974), 163–182.
- [36] D. Voiculescu, K. Dykema and A. Nica, Free random variables, *CRM Monograph Series*, Vol. 1, Amer. Math. Soc., Providence, RI.
- [37] Q. Xu, Applications du théorème de factorisation pour des fonctions à valeurs opérateurs. *Studia Math.* 95 (1990) 273–292.